First Selection Test — Solutions

Problem 1. Given k positive integers $n_1, \cdots, n_k$, define $d_1 = 1$ and

$$d_i = \frac{(n_1, \cdots, n_{i-1})}{(n_1, \cdots, n_i)}, \quad i = 2, \cdots, k,$$

where $(m_1, \cdots, m_\ell)$ denotes the highest common factor of the integers $m_1, \cdots, m_\ell$. Prove that the sums

$$\sum_{i=1}^{k} a_i n_i, \quad a_i \in \{1, \cdots, d_i\}, \quad i = 1, \cdots, k,$$

are pairwise distinct modulo n_1.

Solution. Suppose, if possible, that two of these sums, say $\sum_{i=1}^k a_i n_i$ and $\sum_{i=1}^k b_i n_i$, are congruent modulo n_1 and let j be the largest index such that $a_j \neq b_j$. Notice that $(n_1, \cdots, n_{j-1})$ divides $\sum_{i=1}^{j}(a_i - b_i)n_i$, to deduce that $(n_1, \cdots, n_{j-1})$ divides $(a_j - b_j)n_j$, so d_j divides $(a_j - b_j)n_j/(n_1, \cdots, n_j)$. Finally, notice that d_j and $n_j/(n_1, \cdots, n_j)$ are coprime, to infer that d_j divides $a_j - b_j$, which is a contradiction since $1 \leq |a_j - b_j| < d_j$. The conclusion follows.

Problem 2. Let $ABCD$ be a cyclic quadrangle such that none of the triangles BCD and CDA is equilateral. Prove that, if the Simson line of A in the triangle BCD and the Euler line of this triangle are perpendicular, then so are the Simson line of B in the triangle CDA and the Euler line of this triangle.

Solution. Without loss of generality, we may (and will) assume that the circle $ABCD$ is centred at the origin O, and has unit radius. Let z denote the turn of a point Z. Let M and N be the orthogonal projections of A onto the lines BC and BD, respectively. The line MN is the Simson line of A in the triangle BCD, and

$$m = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right) \quad \text{and} \quad n = \frac{1}{2}\left(a + b + d - \frac{bd}{a}\right). \tag{$*$}$$

Let H be the orthocentre of the triangle BCD. The lines MN and HO are perpendicular if and only if

$$\frac{m-n}{b+c+d} + \frac{\bar{m}-\bar{n}}{\bar{b}+\bar{c}+\bar{d}} = 0.$$

On account of $(*)$, this is the case if and only if $ab + ac + ad + bc + bd + cd = 0$, which is symmetric in a, b, c, d. The conclusion follows.

Problem 3. Given two finite sets A and B of real numbers, and an element x of their Minkowski sum $A + B$, show that

$$|A \cap (x - B)| \leq \frac{|A - B|^2}{|A + B|}.$$

Solution. Rewrite the inequality as

$$|\{(a, b, c) : a \in A, b \in B, c \in A + B, a + b = x\}| \leq |(A - B) \times (A - B)|. \tag{$*$}$$

Next, define an injection of the set on the left-hand side into $(A-B)\times(A-B)$ as follows. Choose, for each $c \in A+B$, elements $a_c \in A$ and $b_c \in B$ such that $c = a_c + b_c$, and assign to each triple (a,b,c) in the set on the left-hand side of $(*)$ the pair $(a-b_c, a_c-b) \in (A-B)\times(A-B)$. Using the identity $c = x-(a-b_c)+(a_c-b)$, it is readily checked that the assignment is injective. The conclusion follows.

Problem 4. Prove that the edges of a planar finite simple graph may be oriented in a such a way that the outdegree of each vertex be at most three.

Solution. Assign an arbitrary orientation to each edge of the graph. The key ingredient is the lemma below.

Lemma. *There is a directed path from any vertex which is not a sink (i.e., of positive outdegree) to some vertex whose outdegree is at most* 2.

Assume the lemma for the moment and let x be a vertex whose outdegree exceeds 3 (if any). By the lemma, there is a directed path from x to some vertex y of outdegree at most 2. Reversal of the orientation of every edge along this path decreases the outdegree of x by 1 and increases that of y by 1, so the latter does not exceed 3. Iteration of this procedure eventually yields an orientation having the desired property.

To prove the lemma, notice that every directed planar graph $G=(V,E)$ has a vertex of outdegree at most 2, for

$$\sum_{x\in V} \operatorname{outdeg} x = |E| \le 3(|V|-2).$$

Now let $x \in V$ have a positive outdegree and let V' be the set of all vertices to which there exists a directed path from x. Clearly, the edges connecting V' and $V \setminus V'$ must enter V', so deletion of these edges makes V' into the vertex set of a directed subgraph G' each vertex of which has the same outdegree as before. Since G' is planar, it has a vertex of outdegree at most 2 and we are through.

Problem 5. Let p and q be coprime positive integer numbers. A $(p+q)$-element set of real numbers $a_1 < a_2 < \cdots < a_{p+q}$ is *balanced* if $a_1, a_2, \cdots, a_p$ form an arithmetic sequence with common difference q, and $a_p, a_{p+1}, \cdots, a_{p+q}$ form an arithmetic sequence with common difference p. Determine the maximum number of balanced $(p+q)$-element sets no two of which are disjoint.

Solution. The required maximum is $p+\max(p,q)$. Notice that two balanced sets have a nonempty intersection, if and only if one is the image of the other through a translation by a number of the form $kp+\ell q$, where $k \in \{0,1,\cdots,q\}$ and $\ell \in \{0,\cdots,p-1\}$. Therefore, the required maximum coincides with the maximum number of distinct integers of the form $kp+\ell q$, $k \in \{0,1,\cdots,q\}$, $\ell \in \{0,\cdots,p-1\}$, whose mutual differences have absolute values of the same form.

We shall prove that the latter maximum is $p+\max(p,q)$. Let $r=\max(p,q)$ and suppose, to the contrary, that there exist $p+r+1$ such integers. Label these integers $k_ip+\ell_iq$, $i=1,2,\cdots,p+r+1$, in lexicographic order: $k_i \le k_{i+1}$, and $\ell_i < \ell_{i+1}$ whenever $k_i = k_{i+1}$.

We first show that there is an index i such that $k_i < k_{i+1}$ and $\ell_i > \ell_{i+1}$. To this end, notice that there are at most q inequalities $k_i < k_{i+1}$, so at least $p+r-q \ge p$ equalities $k_i = k_{i+1}$, hence at least p inequalities $\ell_i < \ell_{i+1}$. This is impossible if $k_i \le k_{i+1}$ and $\ell_i \le \ell_{i+1}$ whatever i, since $0 \le \ell_i \le p-1$.

Now fix an index i such that $k_i < k_{i+1}$ and $\ell_i > \ell_{i+1}$, and recall that $|(k_i p + \ell_i q) - (k_{i+1} p + \ell_{i+1} q)| = kp + \ell q$ for some $k \in \{0, 1, \cdots, q\}$ and some $\ell \in \{0, \cdots, p-1\}$. Explicitly, either $(k_i - k_{i+1})p + (\ell_i - \ell_{i+1})q = kp + \ell q$ or $(k_{i+1} - k_i)p + (\ell_{i+1} - \ell_i)q = kp + \ell q$. Since p and q are coprime, the former implies $\ell_i - \ell_{i+1} \equiv \ell \pmod{p}$, and the latter, $k_{i+1} - k_i \equiv k \pmod{q}$. Taking into account ranges of values, the first congruence forces equality, which in turn leads to a contradiction: $0 > k_i - k_{i+1} = k \geq 0$. In the second case we reach the same contradiction unless $k_{i+1} = q$ and $k_i = k = 0$. So

$$k_j = \begin{cases} 0 & \text{if } j = 1, \cdots, i, \\ q & \text{if } j = i+1, \cdots, p+r. \end{cases}$$

Hence there are at least $(p+r-1)/2$ successive equalities $k_j = k_{j+1}$, so at least as many successive inequalities $\ell_j < \ell_{j+1}$ among the corresponding ℓ's. Since the number of such inequalities among the ℓ's does not exceed $p-1$, it follows that $(p+r-1)/2 \leq p-1$, or $r+1 \leq p$, which is impossible.

Consequently, there are at most $p+\max(p,q)$ integers having the desired property. The examples below show that this is indeed the maximum number of such integers. For more convenience, we exhibit the corresponding k_j and ℓ_j: If $p < q$, the k_j are $0, 1, 2, \cdots, q-1, \underbrace{q, q, \cdots, q}_{p}$, and the ℓ_j are $\underbrace{0, 0, \cdots, 0}_{q+1}, 1, 2, \cdots, p-1$; and if $p > q$, the k_j are $\underbrace{0, 0, \cdots, 0}_{p}, \underbrace{q, q, \cdots, q}_{p}$, and the ℓ_j are $0, 1, 2, \cdots, p-1, 0, 1, \cdots, p-1$. The verifications are straightforward.